

Vector Identity (4)

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$$

Proof

$$\begin{aligned} \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) &= \left(\sum_{i=1}^3 \delta_i A_i \right) \times \left[\left(\sum_{j=1}^3 \delta_j \frac{\partial}{\partial x_j} \right) \times \left(\sum_{k=1}^3 \delta_k B_k \right) \right] \\ &\quad + \left(\sum_{i=1}^3 \delta_i B_i \right) \times \left[\left(\sum_{j=1}^3 \delta_j \frac{\partial}{\partial x_j} \right) \times \left(\sum_{k=1}^3 \delta_k A_k \right) \right] \\ &= \left(\sum_{i=1}^3 \delta_i A_i \right) \times \left[\sum_{j=1}^3 \sum_{k=1}^3 (\delta_j \times \delta_k) \frac{\partial}{\partial x_j} B_k \right] \\ &\quad + \left(\sum_{i=1}^3 \delta_i B_i \right) \times \left[\sum_{j=1}^3 \sum_{k=1}^3 (\delta_j \times \delta_k) \frac{\partial}{\partial x_j} A_k \right] \\ &= \left(\sum_{i=1}^3 \delta_i A_i \right) \times \left(\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_l \varepsilon_{jkl} \frac{\partial B_k}{\partial x_j} \right) \\ &\quad + \left(\sum_{i=1}^3 \delta_i B_i \right) \times \left(\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_l \varepsilon_{jkl} \frac{\partial A_k}{\partial x_j} \right) \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_i \times \delta_l) \varepsilon_{jkl} A_i \frac{\partial B_k}{\partial x_j} \\ &\quad + \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_i \times \delta_l) \varepsilon_{jkl} B_i \frac{\partial A_k}{\partial x_j} \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \delta_m \varepsilon_{ilm} \varepsilon_{jkl} A_i \frac{\partial B_k}{\partial x_j} \\ &\quad + \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \delta_m \varepsilon_{ilm} \varepsilon_{jkl} B_i \frac{\partial A_k}{\partial x_j} \end{aligned}$$

Continue the simplification.

$$\begin{aligned}
\mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) &= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \delta_m \varepsilon_{mil} \varepsilon_{jkl} A_i \frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \delta_m \varepsilon_{mil} \varepsilon_{jkl} B_i \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m (\delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ij}) A_i \frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m (\delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ij}) B_i \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m \delta_{mj} \delta_{ik} A_i \frac{\partial B_k}{\partial x_j} - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m \delta_{mk} \delta_{ij} A_i \frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m \delta_{mj} \delta_{ik} B_i \frac{\partial A_k}{\partial x_j} - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m \delta_{mk} \delta_{ij} B_i \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \delta_{ik} A_i \frac{\partial B_k}{\partial x_j} - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \delta_{ij} A_i \frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \delta_{ik} B_i \frac{\partial A_k}{\partial x_j} - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \delta_{ij} B_i \frac{\partial A_k}{\partial x_j} \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \delta_j A_k \frac{\partial B_k}{\partial x_j} - \sum_{j=1}^3 \sum_{k=1}^3 \delta_k A_j \frac{\partial B_k}{\partial x_j} \\
&\quad + \sum_{j=1}^3 \sum_{k=1}^3 \delta_j B_k \frac{\partial A_k}{\partial x_j} - \sum_{j=1}^3 \sum_{k=1}^3 \delta_k B_j \frac{\partial A_k}{\partial x_j} \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \left(A_k \frac{\partial B_k}{\partial x_j} + B_k \frac{\partial A_k}{\partial x_j} \right) \\
&\quad - \sum_{j=1}^3 A_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^3 \delta_k B_k \right) - \sum_{j=1}^3 B_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^3 \delta_k A_k \right) \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \frac{\partial}{\partial x_j} (A_k B_k) - \sum_{j=1}^3 A_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^3 \delta_k B_k \right) - \sum_{j=1}^3 B_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^3 \delta_k A_k \right) \\
&= \sum_{j=1}^3 \delta_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^3 A_k B_k \right) - \sum_{j=1}^3 A_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^3 \delta_k B_k \right) - \sum_{j=1}^3 B_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^3 \delta_k A_k \right) \\
&= \nabla(\mathbf{A} \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{A}
\end{aligned}$$

Therefore,

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}.$$